Short Papers

Expansions for the Capacitance of the Bowman Squares

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Abstract—**Expansions are given for the capacitance per unit length for the geometry in which two infinite, square cylinders are placed concentric with each other and rotated so that the edges of the inner square are closest to the sides of the outer square.**

I. Introduction

About 50 years ago, Bowman [1] determined the capacitance of the geometry shown in Fig. 1 by means of a conformal transformation which maps the rectangle in the W plane of Fig. 1 onto the quadrilateral, $OABC$, of the Z plane. For the case which he considered explicity (pages 275 and 276) if $\rho = 1 - b/d$,

$$\frac{L}{L'} = \frac{1+(2+i)\rho}{1-i\rho} \tag{1}$$

$$q' = e^{-\pi L/L'} \tag{2}$$

$$\frac{\lambda'}{\lambda} = 4q'^{1/2}\left(\frac{1+q'^2+q'^6+\cdots}{1-2q'+2q'^4+\cdots}\right)^2 \tag{3}$$

$$\frac{ik'}{k} = \left(\frac{1-\lambda'/\lambda}{1+\lambda'/\lambda}\right)^2 \tag{4}$$

$$k'^2 = \frac{\left(i\dfrac{k'}{k}\right)^2}{\left(i\dfrac{k'}{k}\right)^2 - 1} \qquad (\text{i.e., } k^2 + k'^2 = 1) \tag{5}$$

$$\pi\frac{K}{K'} = \log\left(\frac{16}{k'^2}\right) - \frac{1}{2}\left(k'^2 + \frac{13}{32}k'^4 + \frac{23}{96}k'^6 + \cdots\right) \tag{6}$$

$$C_0 = 8\frac{K}{K'}. \tag{7}$$

These equations determine the capacitance, C_0, of the structure explicitly in terms of ρ. The convergence of (3) is rapid for all values of ρ between 0 and 1 since the real part of $L/L' \geqslant 1$ in this range. Care must be taken in the determination of the sign of λ'/λ from (3) to ensure consistency with [1, eq. (6)]. In short, the real part of λ'/λ must be positive when $k > k'$, that is, when $\rho < \sqrt{2} - 1$, and negative when $k < k'$. Equations (1)–(4) involve computations with complex numbers. Otherwise the determination of the capacitance involves real quantities only. The values of C_0 shown in the middle row of Table I were obtained from this sequence of equations.

In this paper, two series for C_0, one in terms of $\delta = 1 - \rho$ and the other in terms of ρ, are given which have certain theoretical and practical advantages over the procedure outlined above. Not only do the series give the limiting behavior of C_0 as ρ approaches 0 or 1 but also the first eight terms in each series permit the direct calculation of C_0 with sufficient accuracy for most engineering applications without requiring any complex algebra.

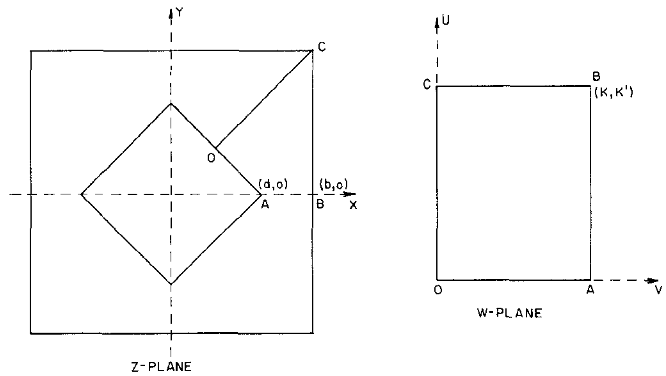

Fig. 1. Z and X coordinate planes.

Those interested in other examples in which the capacitance of symmetrical, doubly connected regions is determined by conformal mapping are referred to Laura and Luisoni [2].

First, consider the simpler case when $0 < k < 1/\sqrt{2}$. If θ is defined by $k = \cos\theta$, this corresponds to $\pi/4 < \theta < \pi/2$ and the evaluation of Z, given by

$$Z = F(\theta)\left\{\int_0^\zeta \frac{d\zeta}{(1-\zeta^2)(1-\lambda^2\zeta^2)} - \int_0^\zeta \frac{d\zeta}{(1-\zeta^2)(1-\lambda'^2\zeta^2)}\right\} \tag{8}$$

is obtained by integration in the left hand ζ plane of Fig. 2. As in [1, eq. (16)], a and b are found from

$$a = \int_0^1 \frac{dZ}{d\zeta}\,d\zeta \qquad ib = \int_1^0 \frac{dZ}{d\zeta}\,d\zeta \tag{9}$$

taken over the proper paths.

The expressions for a and b are not the same as those given in [1, eqs. (16) and (17)] because $1/\lambda$ is now in the upper half plane. It is readily seen from (8) and (9) that

$$a = F(\theta)(L - L' + 2iL') \tag{10}$$

$$ib = F(\theta)(-L - L') \tag{11}$$

where L and L' are the complete elliptic integrals as defined in [1].

If $d = b - a$,

$$\delta = \frac{d}{b} = \frac{(1+i)(L-L')}{L+L'}. \tag{12}$$

This equation is readily solved for L/L' in terms of δ and it is found that

$$\frac{L}{L'} = \frac{1+x}{1-x} \tag{13}$$

if $x = (1-i)\delta/2$.[1]

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[1] This step is crucial in the procedure because our problem is now reduced to considering only series with real coefficients.

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TABLE I
CAPACITANCE OF BOWMAN SQUARES

δ	.1	.2	.3	.4	.5	.6	.7	.8	.9	.95
$C(\delta)$	2.4552	3.3674	4.3032	5.3630	6.6376	8.2588	10.4661	13.7973	19.9590	26.2882
C_o	2.4552	3.3674	4.3032	5.3630	6.6376	8.2588	10.4661	13.7973	19.9769	26.5639
$C(\rho)$					7.2332	8.3697	10.4786	13.7971	19.9769	26.5639
ρ					.5	.4	.3	.2	.1	.05

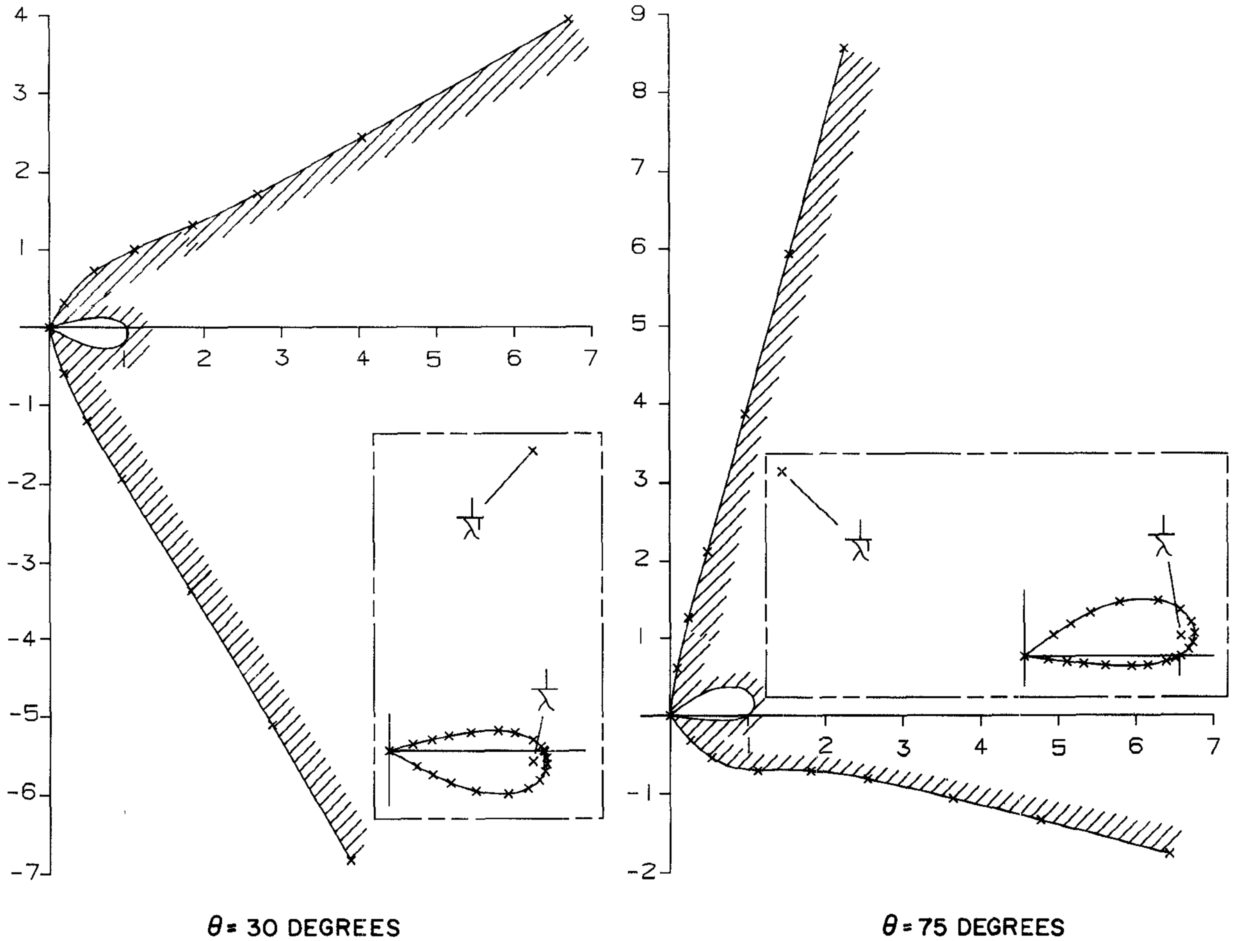

Fig. 2. Accurate ζ plane graphs.

Since (3) requires the expansion of q' and its powers and q' is given by

$$q' = e^{-\pi L/L'} = e^{-\pi} e^{-2\pi x/(1-x)} \tag{14}$$

we are concerned with expansions having the form $\exp(yx/(1-x))$. Of course

$$e^{yx/(1-x)} = 1 + \frac{yx}{1-x} + \frac{y^2x^2}{2!(1-x)^2} + \frac{y^3x^3}{3!(1-x)^3} + \cdots \tag{15}$$

which can be readily shown to be given by

$$e^{yx/(1-x)} = 1 + \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{j} \frac{y^k}{k!} \binom{j-1}{k-1} \right\} x^j. \tag{16}$$

Here the expression bounded by the round parentheses is the number of combinations of $j-1$ things taken $k-1$ at a time. Then

$$q' = e^{-\pi} \left[1 + \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{j} \frac{(-1)^k (2\pi)^k}{k!} \binom{j-1}{k-1} \right\} x^j \right] \tag{17}$$

and

$$q'^n = e^{-n\pi} + \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{j} \frac{(-1)^k (2n\pi)^k e^{-n\pi}}{k!} \binom{j-1}{k-1} \right\} x^j \tag{18}$$

so that

$$q'^{1/4} = e^{-\pi/4} + \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{j} \frac{(-1)^k (\pi/2)^k e^{-\pi/4}}{k!} \binom{j-1}{k-1} \right\} x^j. \tag{19}$$

Moreover

$$1+q'^2+q'^6+q'^{12}+\cdots$$

$$=\sum_{l=0}^{\infty} q'^{(l^2+l)} \quad (20)$$

$$=1+\sum_{l=1}^{\infty}\left[e^{-\pi(l^2+l)}+\sum_{j=1}^{\infty}\left\{\sum_{k=1}^{J}\left[2\pi(l^2+l)\right]^k \cdot e^{-\pi(l^2+l)}\frac{(-1)^k}{k!}\binom{j-1}{k-1}\right\}\right]x^J \quad (21)$$

$$=\sum_{l=0}^{\infty} e^{-\pi(l^2+l)}+\sum_{j=1}^{\infty}\left\{\sum_{k=1}^{J}\left[\sum_{l=1}^{\infty}\left[2\pi(l^2+l)\right]^k e^{-\pi(l^2+l)}\right] \cdot \frac{(-1)^k}{k!}\binom{j-1}{k-1}\right\}x^J \quad (22)$$

after an interchange in the order of summation.

On the other hand,

$$1-2q'+2q'^4-2q'^9+\cdots$$

$$=1+2\sum_{l=1}^{\infty}(-1)^l q'^{l^2} \quad (23)$$

$$=1+2\sum_{l=1}^{\infty}\left[(-1)^l e^{-\pi l^2}+(-1)^l \cdot \sum_{j=1}^{\infty}\left\{\sum_{k=1}^{J}(2\pi l^2)^k e^{-\pi l^2}\frac{(-1)^k}{k!}\binom{j-1}{k-1}\right\}\right]x^J \quad (24)$$

$$=1+2\sum_{l=1}^{\infty}(-1)^l e^{-\pi l^2} + \sum_{j=1}^{\infty}\left\{\sum_{k=1}^{J}\left[\sum_{l=1}^{\infty}(-1)^l(2\pi l^2)^k e^{-\pi l^2}\right] \cdot \frac{(-1)^k}{k!}\binom{j-1}{k-1}\right\}x^J. \quad (25)$$

The expansion of the two series in q', in terms of x, depends on the convergence of the exponential series in the square brackets. The value of k occurring in a given coefficient is limited by the exponent, j, of the term in question. Thus, after some value of l, depending on k, the exponential series in the square brackets converge rapidly. Then since the sum in k is finite, the values of the coefficients in (22) and (25) can be determined with an accuracy limited only by the computer being used. Then, carrying out the steps indicated in (3), (4), and (5), it was found that the expansion for the modulus squared, k^2, involves only[2] powers of x^4 since k^2 is a real function of δ. Thus no computation with complex numbers is required.

Finally,

$$k^2 = 5.734272\delta^4 - 15.344874\delta^8 + 26.696796\delta^{12} - 35.595891\delta^{16} + 39.859610\delta^{20} - 39.271437\delta^{24} + 35.016388\delta^{28} - 28.822722\delta^{32} + 22.199578\delta^{36} - 16.164546\delta^{40} + 11.214410\delta^{44} - 7.458393\delta^{48} + \cdots. \quad (26)$$

[2]The fact that the values of the coefficients of the other powers of x were less than 10^{-10} gives an indication of the accuracy of the calculations.

Since, from the Appendix,

$$\pi\frac{K'}{K} = \log\left(\frac{16}{k^2}\right) - \frac{1}{2}\left[k^2 + \frac{13}{32}k^4 + \frac{23}{96}k^6 + \frac{2701}{16384}k^8 + \frac{5057}{40960}k^{10} + \frac{76715}{786432}k^{12} + \frac{146749}{1835008}k^{14} + \cdots\right] \quad (27)$$

substitution of (26) in (27) gives, in view of (7),

$$C_0 = 8\pi/[4\log(1/\delta) + 1.026128 - 0.191142\delta^4 - 0.081878\delta^8 - 0.052318\delta^{12} - 0.037011\delta^{16} - 0.027610\delta^{20} - 0.021508\delta^{24} - 0.018969\delta^{28} + \cdots]. \quad (28)$$

As will be shown, this expansion gives accurate values of C_0 for values of δ as high as 0.8.

To obtain an expansion for small values of ρ, we return to equation (1). Then

$$\frac{L}{L'} = 1 + \frac{2(1-i)x}{1-x}. \quad (29)$$

q' is now given, as before, by (14), (15), and (16) except that $y=-2\pi(1-i)$. Accordingly equations for $q'^{1/4}$, $1+q'^2+1'^6+q'^{12}+\cdots$, and $1-2q'+2q'^4-2q'^9+\cdots$ are identical to (19), (22), and (25) except for an additional factor, $1-i$, inside of the small parentheses which is raised to the kth power. For example,

$$1+q'^2+q'^6+q'^{12}+\cdots$$

$$=\sum_{l=0}^{\infty} q'^{(l^2+l)} \quad (30)$$

$$=\sum_{l=0}^{\infty} e^{-\pi(l^2+l)}+\sum_{j=1}^{\infty}\left\{\sum_{k=1}^{J}\left[\sum_{l=1}^{\infty}\left[2\pi(l^2+l)\right]^k e^{-\pi(l^2+l)}\right] \cdot \frac{(i-1)^k}{k!}\binom{j-1}{k-1}\right\}x^J. \quad (31)$$

If we then put $u(j,k)+iv(j,k)=\frac{(i-1)^k}{k!}\binom{j-1}{k-1}$, we determine $\sqrt{\lambda'/\lambda}$ as the sum of a real series in x plus i times a second real series in x without making any computations with complex numbers except for determining powers of $i-1$. On evaluation, it is found that the real component of k' is an even series in x while the imaginary component is an odd series in x. Consequently when x is replaced by $i\rho$ a real series in ρ results.

Finally,

$$k'^2 = 91.748350\rho^4 - 366.993400\rho^5 + 917.483500\rho^6 - 1834.966999\rho^7 + 717.095621\rho^8 - 26288.395363\rho^9 + 133711.501930\rho^{10} - 460384.742429\rho^{11} + \cdots \quad (32)$$

and

$$C_0 = \frac{8}{\pi}[-4\log(\rho) - 1.746461 + 4\rho - 2\rho^2 + 1.333333\rho^3 - 4.058278\rho^4 + 13.033113\rho^5 - 31.249450\rho^6 + 61.736995\rho^7]. \quad (33)$$

Table I compares the exact value of the capacitance of the Bowman squares with the values obtained from the two approximate expansions. The upper row gives the values obtained from (28). It is seen that these values agree with the exact values given in the middle row to six significant places for δ as great as

0.8. In fact, the error is less than 1 percent for $\delta = 0.95$. On the other hand, (33) is accurate for small values ρ. These expansions also give the limiting behavior of the capacitance as δ and ρ approach zero.

APPENDIX

The expansion for C_0 in terms of δ given in (28) requires more terms in the expansion for K'/K in terms of k than are generally known. For future possible interest, the first 12 terms in this expansion are

$$\pi\frac{K'}{K} = \log\left(\frac{16}{k^2}\right) - \frac{1}{2}\left[k^2 + \frac{13}{32}k^4 + \frac{23}{96}k^6 + \frac{2701}{16384}k^8 + \frac{5057}{40960}k^{10} + \frac{76715}{786432}k^{12} + \frac{146749}{1835008}k^{14} + \frac{144644749}{2147483648}k^{16} + \frac{279805685}{4831838208}k^{18} + \frac{4346533901}{85899345920}k^{20} + \frac{8465644159}{188978561024}k^{22} + \cdots\right]. \quad (34)$$

The method used to determine these coefficients is detailed in Riblet [3, p. 665]. The identity [4, p. 73]

$$\frac{K(k_0)}{K'(k_0)} = 2\frac{K(k_1)}{K'(k_1)} \quad (35)$$

when

$$k_0 = \frac{2\sqrt{k_1}}{1+k_1}$$

can also be used to find them as the solution of a set of linear equations if an expansion for K'/K of the required form is substituted in (35). In fact, it is the complete agreement between the decimal values obtained from these independent procedures which provides the author with the confidence to present these computer-generated rational values.

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A Coplanar Probe to Microstrip Transition

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***Abstract*—A transition between a coplanar probe and a microstrip transmission line is reported. The transition is significant in that it does not require substrate via holes. A set of microstrip impedance standards were developed for the purpose of de-embedding the transition. The transition is suitable for measuring the S parameters of a number of low-cost monolithic microwave integrated circuits with coplanar probes.**

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I. INTRODUCTION

The S parameters of planar microwave devices may be measured accurately and efficiently with coplanar probes [1]. Using coplanar probes to measure the S parameters of monolithic microwave integrated circuits (MMIC's) based on microstrip transmission lines is difficult, however, because the ground plane of the microstrip transmission lines is not easily contacted by the coplanar probe. Moniz [2] and Harvey [3] have used plated substrate via holes for this purpose, but the difficult processing required to form these via holes inevitably raises the circuit fabrication cost and lowers the yield. For these reasons it is often desirable to use a transition which does not require substrate via holes, especially if the circuit to be fabricated does not require via hole grounding.

In this work a transition between a coplanar probe and a microstrip transmission line which does not require substrate via holes and which is suitable for performing S parameter measurements of microstrip MMIC's is described. The method used to de-embed and unterminate the transition is discussed and the S parameters of the transition deduced from this procedure are presented. The accuracy of the de-embedded measurements is estimated.

II. COPLANAR PROBE TO MICROSTRIP TRANSITION

The coplanar probe to microstrip transition investigated in this work is shown in Fig. 1. The center signal pad on the coplanar probe contacts the microstrip line into which the signal is launched. The outer two ground pads on the coplanar probe contact the microstrip radial stub near its center. The microstrip radial stub provides a low impedance between the microstrip ground plane and the coplanar probe ground contacts. (Other stub types could be used but would exhibit a more narrow band performance.)

If the fringing fields at the stub edges can be ignored, the electrical reactance X between the probe ground and the microstrip ground can be estimated from the formulas given by Atwater [4]:

$$X = (h/2\pi r_1)\,Z_0(r_1)(360/\theta)\cos(\vartheta_1 - \Phi_2)/\sin(\Phi_1 - \Phi_2) \quad (1)$$

where

$$\tan(\theta_1) = N_0(kr_1)/J_0(kr_1)$$

$$\tan(\Phi_i) = -J_1(kr_i)/N_1(kr_i) \qquad (i = 1,2)$$

$$Z_0(r_1) = (120\pi/\sqrt{\epsilon_{re}})\left[J_0^2(kr_1) + N_0^2(kr_1)\right]^{1/2} \cdot \left[J_1^2(kr_1) + N_1^2(kr_1)\right]^{-1/2}$$

$$k = 2\pi\sqrt{\epsilon_{re}}/\lambda_0$$

and θ is the angle subtended by the stub, ϵ_{re} is the effective dielectric constant (which may be approximated by the substrate relative dielectric constant ϵ_r for large θ), h is the substrate thickness, λ_0 is one free-space wavelength, $J_i(x)$ and $N_i(x)$ are